$$a\int_{\frac{1}{3}a}^{\frac{1}{2}a}\int_{\frac{1}{2}(3x-a)}^{\frac{1}{4}a} dxdy = \tfrac{3}{4}ax^2 = u_{11} = \tfrac{1}{48}a^3.$$

As the whole number of different trials that can be made will be represented by $8a^3$, if P represent the required probability, because the stable positions above calculated are just half the whole number of stable positions, (x having been taken only from the center of (1) to *one* extremity,) we shall have

$$P = \frac{u_1 + u_2 + \ldots\ldots + u_{11}}{4a^3} = \frac{209}{2304}.$$

SOLUTION OF TWO PROBLEMS IN SUMMATION OF SERIES.

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1. REQUIRED the sum of the series

$$1 + \frac{1}{1+r} + \frac{1}{1+2r} + \frac{1}{1+3r} + \ldots + \frac{1}{1+(n-1)r} = \Sigma_n^{(r)} \ldots\ldots (1)$$

Solution.—First find the value of the definite integral

$$\frac{1}{r}\int_{-1}^{0} y^p dy(1+y)^{\frac{1}{r}-1} = \Big[_{-1}^{0} y^p(1+y)^{\frac{1}{r}} - \frac{py^{p-1}r(1+y)^{(1\div r)-1}}{1+r}$$
$$+ \frac{p(p-1)y^{p-2}r^2(1+y)^{(1\div r)+2}}{(1+r)(1+2r)} + (-1)^p.\frac{p(p-1)\ldots 2.1.r^p(1+y)^{(1\div r)+p}}{(1+r)(1+2r)\ldots(1+pr)}\Big]$$
$$= (-1)^p.\frac{p(p-1)\ldots 2.1.r^p}{(1+r)(1+2r)\ldots(1+pr)} = V_p \ldots . (2)$$

Make $p = 0, 1, 2, 3,$ &c., successively, and we shall have

$$V_0 = 1,\ V_1 = -\frac{r}{1+r},\ V_2 = \frac{2r^2}{(1+r)(1+2r)},$$
$$V_3 = -\frac{3.2r^3}{(1+r)(1+2r)(1+3r)},\ \text{\&c.} \ldots .(3)$$

Now take the series

$$1 + x^r + x^{2r} + \ldots . + x^{(n-1)r} = \frac{x^{nr}-1}{x^r-1}, \qquad \text{then}$$

$$\Sigma_n^{(r)} = \int_0^1 dx(1 + x^r + x^{2r} + \ldots) = \int_0^1 dx.\frac{x^{nr}-1}{x^r-1}.$$

Put $x^r = 1 + y$, $x = \sqrt[r]{(1+y)}$, $dx = (1\div r)\times \sqrt[r-1]{(1+y)}dy$, and

$$\frac{x^{nr}-1}{x^r-1} = \frac{(1+y)^n-1}{y} = n + \frac{n(n-1)y}{1\,.\,2} + \frac{n(n-1)(n-2)y^2}{1\,.\,2\,.\,3} + \ldots$$

$$\Sigma_n^{(r)} = \int_{-1}^{0} \frac{n}{r}\left[(1+y)^{\frac{1}{r}-1} + \frac{(n-1)y}{1.2}(1+y)^{\frac{1}{r}-1} + \&c.\right]dy. \therefore \text{ by (2)\&(3)}$$

$$\Sigma_n^{(r)} = n - \frac{n(n-1)r}{1.2(1+r)} + \frac{n(n-1)(n-2)r^2}{1.2.3.(1+r)(1+2r)} - \frac{n(n-1)(n-2)(n-3)r^3}{1.2.3.4.(1+r)(1+2r)(1+3r)} + \dots \quad (4)$$

$$\Sigma_n^{(1)} = n - \frac{n(n-1)}{1.2.2} + \frac{n(n-1)(n-2)}{1.2.3.2.3} - \frac{n(n-1)(n-2)(n-3)}{1.2.3.4.2.3.4} + \dots \quad (5)$$

From (4) we easily find

$$\Sigma_{n+1}^{(r)} - \Sigma_n^{(r)} = \frac{1}{1+nr} = 1 - \frac{nr}{1+r} + \frac{n(n-1)r^2}{1.2(1+r)(1+2r)} - \frac{n(n-1)(n-2)r^3}{1\cdot 2.3(1+r)(1+2r)(1+3r)} + \dots$$

$$\frac{1}{1-nr} = 1 + \frac{nr}{1+r} + \frac{n(n-1)r^2}{1.2.(1+r)(1+2r)} + \frac{n(n-1)(n-2)r^3}{1.2.3(1+r)(1+2r)(1+3r)} + \dots$$

$$\frac{1}{1-n^2r^2} = 1 + \frac{n(n-1)r^2}{1.2(1+r)(1+2r)} + \frac{n(n-1)(n-2)(n-3)r^4}{1.2.3.4(1+r)(1+2r)(1+3r)(1+4r)} + \dots$$

2. Find the sum of the series

$$S_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots + \frac{1}{n^2} \dots \dots \quad (6)$$

Since $1 + x + x^2 + x^3 + \dots + x^{n-1} = x^{n-1} \div (x-1)$ we evidently have

$$S_n = \int_0^1 \frac{dx}{x} \int_0^x dx\, \frac{x^n - 1}{x - 1} \dots \dots \quad (7)$$

Put $x - 1 = y$, $dx = dy$, then

$$\frac{x^n - 1}{x - 1} = \frac{(1+y)^n - 1}{y} = n + \frac{n(n-1)y}{1.2} + \frac{n(n-1)(n-2)y^2}{1.2.3} + \dots$$

$$\int_0^x dx\, \frac{x^n - 1}{x - 1} = \int_{-1}^{y} dy\left[n + \frac{n(n-1)y}{1.2} + \dots\right] = ny + \frac{n(n-1)}{1.2.2}y^2 + \frac{n(n-1)(n-2)}{1.2.3.3}y^3 + \dots + n - \frac{n(n-1)}{1.2.2} + \frac{n(n-1)(n-2)}{1.2.3.3} + \dots$$

$$= n(y+1) + \frac{n(n-1)}{1.2.2}(y^2-1) + \frac{n(n-1)(n-2)}{1.2.3.3}(y^3+1) + \dots$$

Therefore, since $dx \div x = dy \div (1+y)$

$$S_n = \int_{-1}^{0} dy\left[n + \frac{n(n-1)}{1.2.2}(y-1) + \frac{n(n-1)(n-2)}{1.2.3.3}(y^2-y+1) + \dots\right] = n - \frac{n(n-1)}{1.2.2}(1+\tfrac{1}{2}) + \frac{n(n-1)(n-2)}{1.2.3.3}(1+\tfrac{1}{2}+\tfrac{1}{3}) - \frac{n(n-1)(n-2)(n-3)}{1.2.3\cdot 4.4}(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}) + \dots$$

$$\therefore S_n = n - \frac{n(n-1)}{1.2.2}\Sigma_2^{(1)} + \frac{n(n-1)(n-2)}{1.2.3.3}\Sigma_3^{(1)} - \frac{n(n-1)(n-2)(n-3)}{1.2.3.4.4}\Sigma_4^{(1)} + \dots \quad (8)$$